On formulas of the Lannes and Viro-Polyak type for finit degree invariants.

(It is very brief text. More complete version of this paper is in Mat.Zametki **t.65** N6 (1998).)

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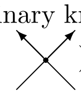
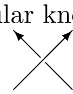
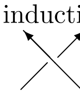
Abstract

Vassiliev's knot invariants can be computed in different ways but many of them as Kontsevich integral are very difficult. We consider more visual diagram formulas of the type Polyak-Viro and give new diagram formula for the two basic Vassiliev invariant of degree 4.

1 The Vassiliev knot invariants.

Let $K : S^1 \longrightarrow \mathbf{R^3}$ be an oriented knot and $K_n^{\text{sing}} : S^1 \longrightarrow \mathbf{R^3}$ be a singular knot with n double points. Denote by $\mathcal{K}$ the space of knots and by $\mathcal{K}^{\text{sing}}$ the space of singular knots. Any knot invariant $V : \mathcal{K} \longrightarrow \mathbf{Q}$ may be extended from ordinary knots to singular knots by next inductive rule:

$V^{(i)}($  $) = V^{(i-1)}($  $) - V^{(i-1)}($  $).$ (1)

Definition. A knot invariant $V : \mathcal{K} \longrightarrow \mathbf{Q}$ is said to be Vassiliev invariant of degree less than or equal to n, if there exists $n \in \mathbf{N}$ such that

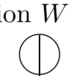
$$V^{(n+1)} \equiv 0.$$

2 Weight systems.

For any singular knot we construct it's chord diagram. A chord diagram D_n of the singular knot K^{sing} is the circle with pre-images of double points connected with chords.

Denote by $\mathcal{D}_n$ the free abelian group generated by diagrams with n chord. For any invariant f with values in an abelian group A there exists a homonorphism $W_f : \mathcal{D}_n \longrightarrow A$, given by $W_f = f|_{\mathcal{K}_n^{sing}}$, where $\mathcal{K}_n^{\text{sing}}$ is the space of knots with n double points.

Definition. A linear function W is called a weight system of degree n if it satisfies next relations:

1-term relation: $W_n($  $) = 0$

(this diagram has n chords, one of which is isolated) and

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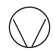
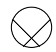
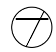
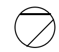
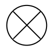
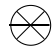
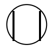
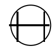
4-term relation: $W_n($  $) - W_n($  $) + W_n($  $) - W_n($  $) = 0$

these diagrams have n chords, $(n-2)$ of which are not drown here and 2 chords are positioned as shown.

Defineded above homonorphism W_f is a weight system.

Examples 1. Weight systems of degrees 2 and 3.

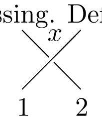
$W_2($  $)=1$ $\qquad$ $W_3($  $)=2$

$W_2($  $)=0$ $\qquad$ $W_3($  $)=1$

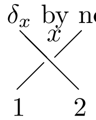
$W_3 = 0$ in other cases.

3 "Coordinates" on knots.

Let D be a diagram of knot K and x be a double point of D. Numerate branches in neighbourhood of x according to the order of their passing. Define function δ_x by next rule:



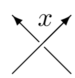
$\delta_x = 0$



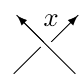
$\delta_x = 1$

(the orientations of branches are not important).

Define function ε_x as follows:

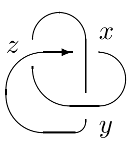


$\varepsilon_x = +1$



$\varepsilon_x = -1$

Examples 2. "Coordinates" on trefoil.



$\delta_x = 1, \quad \varepsilon_x = 1$

$\delta_y = 0, \quad \varepsilon_y = 1$

$\delta_z = 1, \quad \varepsilon_z = 1$

4 Formulas of Lannes.

For the basic Vassiliev invariants of degree 2 and 3, which take values 0 on trivial knot and 1 on trefoil we have next formulas:

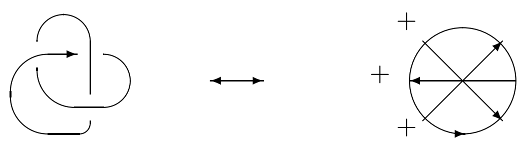
$$V_2(K) = 1/2 \sum_{\{x,y\}\in P_2} (-1)^{\delta_x+\delta_y} W_2(\{x,y\})\varepsilon_x\varepsilon_y[\delta_x(1-\delta_y)+\delta_y(1-\delta_x)],$$

$$V_3(K) = 1/2 \sum_{\{x,y,z\}\in P_3} (-1)^{\delta_x+\delta_y+\delta_z} W_3(\{x,y,z\})\varepsilon_x\varepsilon_y\varepsilon_z[\delta_y(1-\delta_x)(1-\delta_z)-\delta_x\delta_z(1-\delta_y)],$$

where the sum is taken over all unordered pairs (triplets) of double points of planar projection, $W_2(\{x,y\})$ ($W_3(\{x,y,z\})$) is weight of chord diagram corresponding to pair (triplet) of double points.

5 The Gauss diagrams.

A chord diagram of the singular knot is the circle with pre-images of double points connected with chords. To obtain the analogus diagram of an ordinary knot (that is called an arrow diagram) from the chord diagram of corresponding singular knot we must give the information on overpasses and underpasses. Each chord is oriented from the upper branch to the lower one and equipped with the sign (the local writh number of corresponding double point of planar projection of the knot).



K G

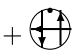
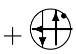
6 Formulas of Viro-Polyak.

Denote by $< A, G >$ algebraic number of subdiagrams of given combinatorial type $A \quad A \subset G$ and let $< \sum_i n_i A_i, G >= \sum_i n_i < A_i, G >, \quad n_i \in \mathbf{Q}$ by definition. Then

$V_2(K) =<$  $, G >,$

$V_3(K) =< [$  $] + \frac{1}{2}[$  $], G > .$

Theorem. *Let v_3 be basic Vassiliev invariant of degree 3, which take values 0 on trivial knot and 1 on trefoil we have next formula:*

$v_3(K) =<$  $+$  $+$  $+$  $+$  $, G > .$

Proof see in [5].

The Vassiliev module. The Vassiliev module of degree n is a module over $\mathbf{Q}$ generated by isotopies classes of oriented knots and singular knots with next relations:
1. E=0, where E is the trivial knot,
2. the Vassiliev skein-relation (1),
3. $K_m^{sing} = 0$, if $m > n$.

Theorem. Any knot K in the Vassiliev module of degree $n \quad (n > 1)$ has following expansion:

$$K = \sum_{i=1}^{r+s} v_i(K) K_i,$$

where r is the dimension of the space of the Vassiliev invariants of degree less than or equal to $(n-1)$,
s is the dimension of the space of weight system of degree n,
v_i is the Vassiliev invariants of degree less than or equal to n,
K_i is fixed basic knots. Examples.

$n = 2$ (J.Lannes)

$$K = V_2(K)T$$

$n = 3$ (J.Lannes)

$$K = V_3(K)[T + H] - V_2(K)H$$

$n = 4$

$$K = [V_2/2 + V_3/2 - 3V_4^1 + 4V_4^2 + V_4^3](K)T + [V_2/2 - V_3/2 + V_4^3](K)T^* +$$
$$[-V_4^1 + V_4^2 + 2V_4^3](K)H + [V_4^1 - 3V_4^2](K)F + V_4^2(K)P$$

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